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# EXPONENTIAL STABILITY OF REGULAR LINEAR SYSTEMS ON BANACH SPACES

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**Abstract:** The article deals with the vd-transformation in Banach space and its application in studying the stability of trivial solution of differential equations. A sufficient condition of exponential stability of regular linear systems with burfication on Banach space will be proved.

## vd-transformation and it's properties

In this section we shall give the definition, examples and some properties of a vd-transformation on Banach spaces. It is an expansion of a vd-transformation on finite dimension spaces given by Yu. S. Bogdanov ([2]–[6]). From that, we shall give the definition of regular linear equations which are applied to study the stability of regular linear equations with burfication on Banach spaces.

Let $E$ be a Banach space and $G$ be an open simple connected domain containing the origin $O$ of $E$.

We define $H$ as follows $H = G \times \mathbf{R} = \{\eta = (x,t)\colon\ x \in G,\quad t \in \mathbf{R}\}$.

Let $v_0\colon \mathbf{R}^+ \to \mathbf{R}^+$ be a function which is continuous, monotone strictly increasing and satisfies the following conditions:

$$v_0(0) = 0; \quad v_0(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Let $d\colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$ be a given real function of two variables: and $d$ satisfies the following conditions for all $\gamma > 0, \gamma_3 > \gamma_2 > \gamma_1 > 0$;

($\mathrm{d}_1$) $d(\gamma_2,\gamma_1) = -d(\gamma_1,\gamma_2)$,

($\mathrm{d}_2$) $d(\gamma_2,\gamma) > d(\gamma_1,\gamma)$,

($\mathrm{d}_3$) $d(\gamma_3,\gamma_2) + d(\gamma_2,\gamma_1) \geq d(\gamma_3,\gamma_1)$,

($\mathrm{d}_4$) $\cup_{\gamma\in\mathbf{R}_+}\{d(\gamma,\gamma_1)\} = \mathbf{R}$.

Suppose that, $l\colon H \to H$ is a diffeomorphism,

$$\eta = (x,t) \mapsto \eta' = (x',t')$$

satisfying the following equalities:

$$l(0,t) = (0,t), \qquad l(x,t) = (x',t)$$

for all $t \in \mathbf{R}$. It is easy to prove that the set $L$ of all those transformations $L = \{l\}$ is a group with the composition of maps.

Let $v$ be a real function

$$v\colon H^\star \to \mathbf{R}_+, \quad \eta = (x,t) \to v(\eta) = v_0(\|x\|)$$

where $H^\star = G^\star \times \mathbf{R} = (G \setminus \{0\}) \times \mathbf{R}$.

Since the function $v\colon H^\star \to \mathbf{R}_+$ is independent of $t$, that is, $v(x,t) = v(x,t')$ for all $t, t' \in \mathbf{R}$, we can denote by $v(x)$ the value of $v(x,t)$ for any $x \in G^\star$ and $t \in \mathbf{R}$.

**Definition.** The transformation $l \in L$ is called vd-transformation iff

$$\sup_{\eta \in H^\star} |d\{v(\eta), v[l(\eta)]\}| < +\infty \tag{1}$$

From the definition of function $d$, we also have

$$\sup_{\eta' \in H^\star} \left| d\left\{ v(\eta'), v(l^{-1}(\eta')) \right\} \right| < +\infty.$$

Consequently, if we denote by $L_{vd}$ the set of vd-transformation then it is a subgroup of $L$.

**Examples**

1. Suppose $v_0(x,t) = \|x\|, d_0(\gamma_1, \gamma_2) = ln\frac{\gamma_1}{\gamma_2}$, and $l(x,t)$ (with a fixed $t$) is a linear transformation having bounded partial derivation with respect to $t$. Then, $l$ is $v_0 d_0$-transformation if and only if it's a Lyapunov transformation ([1]).
2. If $v(x,t) = |x|^2; E = \mathbf{R}$

$$d(\gamma_1, \gamma_2) = \begin{cases} \sqrt{\gamma_1} - \sqrt{\gamma_2} & \text{if } \gamma_1 \cdot \gamma_2 \geq 1 \\ \frac{1}{\sqrt{\gamma_2}} - \frac{1}{\sqrt{\gamma_1}} & \text{if } \gamma_1 \cdot \gamma_2 < 1, \end{cases}$$

then all conditions from $\mathrm{d}_1$ are satisfied. So

$$l(x,t) = (x + \frac{1}{2} \sin t \sin^2 x, t)$$

is a vd-transformation.

From example 1, we can see that a vd-transformation is an expansion of Lyapunov transformation, but it still keeps an important property, stability of the trivial solution of a following differential equation on the Banach space $E$:

$$\begin{cases} \frac{dx}{dt} = f(x,t) \\ f(0,t) \equiv 0 \end{cases} \tag{2}$$

We denote by $x(t;\xi)$ the solution of equation (2) which satisfies the initial condition $x(t_0,\xi)=\xi$ and suppose that

$$\lambda = \lim_{\varepsilon\to 0^+} \sup_{\substack{\|\xi\|\le\varepsilon\\ t\ge t_0}} \|(x(t;\xi)\|; \lambda_1 = \lim_{\varepsilon\to 0^+} \sup_{\varepsilon\to 0^+} \sup_{\substack{v(\xi)\le\varepsilon\\ t\ge t_0}} v(x(t;\xi)).$$

**Definition.** ([7]). The solution $x=0$ of differential equation (2) is said to be Lyapunov stable if for any $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for each solution $x(t)$ of (2), with its initiala value $x(t_0)=\xi$ satisfying the condition $\|\xi\|<\delta(\varepsilon)$

then the inequality $\|x(t,\xi)\|<\varepsilon$ holds for all $t\ge t_0$.

From the definition we can see that the solution $x=0$ of differential equation (2) is stable iff $\lambda=0$.

**Proposition 1.** $\lambda=0$ *if and only if* $\lambda_1=0$.

**Proof.** By the continuity of the function $v$ we immediately have $\lim\limits_{\xi\to 0} v(\xi)=0$. Since $v(\|x\|)$ is monotone strictly increasing we can deduce $\lim\limits_{v(\xi)\to 0} \xi=0$. Therefore

$$\lim_{k\to\infty} \xi_k = 0 \Leftrightarrow \lim_{k\to\infty} v(\xi_k)=0. \tag{3}$$

We assume that $\lambda=0$, then:

$$\lim_{k\to\infty} \|x(t_k,\xi_k)\| = 0$$

for all sequences $\{\varepsilon_k\}\subset \mathbf{R}_+ :\ \varepsilon_k\to 0; \{\xi_k\}\subset E:\ \|\xi_k\|<\varepsilon_k$ and $\{t_k\}\subset\mathbf{R}, t-k\ge t_0$. Because of (3), we have

$$\lim_{k\to\infty} \|x(t_k,\xi_k)\| = 0 \Leftrightarrow \lim_{k\to\infty} v\left(x\left(t_k,\xi_k\right)\right)=0.$$

It follows that $\lambda=0\Leftrightarrow\lambda_1=0$.

**Proposition 2.** *A vd-transformation conserves the stability of trivial solution* $x=0$ *of differential equation (2).*

**Proof.** By the vd-transformation

$$(x,t)\to l(x,t)=(y,t),$$

the equation (2) is transformed to the following one:

$$\frac{dy}{dt}=g(y,t) \tag{4}$$

By assumption, the solution $x = 0$ of equation (2) is stable, that means:

$$\lim_{\varepsilon\to 0^+} \sup_{\substack{\|x_0\|\le\varepsilon \\ t\ge t_0}} \|x(t;x_0)\| = 0 \Leftrightarrow \lim_{\varepsilon\to 0^+} \sup_{\substack{v(x_0)\le\varepsilon \\ t\ge t_0}} v[x(t;x_0)] = 0.$$

If the solution $y = 0$ of (4) is unstable, then

$$\lim_{\varepsilon\to 0^+} \sup_{\substack{v(y_0)\le\varepsilon \\ t\ge t_0}} v\left[y\left(t, y_0\right)\right] > 0.$$

It means that there exists a positive number $\delta$ such that

$$\text{(5)} \qquad \exists\{\eta_n\} \subset E:\ \eta_n \to 0;\ \exists\{t_n\} \subset \mathbf{R}_+:\ t_n \ge t_0;\ \forall nN:\ v\left[y\left(t_n;\eta_n\right)\right] \ge \delta.$$

Since $v\left[x\left(t_n,\xi_n\right)\right] \to 0$ as $n \to \infty$, where $\xi_n, t_n) = l^{-1}(\eta_n, t_n)$ one could say

$$\text{(6)} \qquad v\left[x\left(t_n;\xi_n\right)\right] < \delta, \forall n \in N.$$

From (5), (6) and $d_4$) we deduce:

$$\left|d\{v\left[x\left(t_n;\xi_n\right)\right], v\left[y\left(t_n;\eta_n\right)\right]\}\right| = d\{v\left[y(t_n;\eta_n)\right], v\left[x\left(t_n;\xi_n\right)\right]\}$$

$$> d\{\delta, v\left[x\left(t_n;\xi_n\right)\right]\} \to +\infty \quad \text{as} \quad n \to \infty.$$

Consequently

$$\sup_{n\in N} \left|d\{v\left[x\left(t_n;\xi_n\right)\right], v\left[l\left(x\left(t_n;\xi_n\right)\right)\right]\}\right| = +\infty$$

that contradicts the definition of 1.

## Regular system

**Definition.** A transformation $l \in L$, satisfying the following condition for all $\eta \in H^\star$ :

$$d\{v\left(\eta\right), v\left[l\left(\eta\right)\right]\} = o(t) \quad \text{as} \quad t \to \pm\infty,$$

is called a generalized vd-transformation.

**Definition.** A transformation $y = L(t)x$ is a generalized Lyapunov one if:

$$\text{(7)} \qquad \chi\left[L\left(t\right)\right] = \chi\left[L^{-1}\left(t\right)\right] = 0$$

where $\chi\left[L\left(t\right)\right] := \overline{\lim_{t\to\infty}} \frac{1}{t} ln\|L(t)\|$ is called characteristic exponent of L(t).

By definition we immediately have following remarks:

**Remark 1.** Generalized Lyapunov transformations conserve Lyapunov exponents [1].

**Remark 2.** A generalized Lyapunov transformation is generalized vd-transformation when

$$v(x) = \|x\|, d(\gamma_1, \gamma_2) = ln\frac{\gamma_1}{\gamma_2},$$

and $l$ is homogeneously linear for $x$ (where $l(x,t) = (L(t)x, t)$).

Now we shall prove a necessary and sufficient condition for which a differential system on finite dimension spaces are regular. Since this condition plays an important role for the conception of a regular differential equations on Banach spaces and we could not find it in literature, we shall formulate it as a lemma.

We consider the following linear differential system:

$$\frac{dx}{dt} = A(t)x \tag{8}$$

where $x \in \mathbf{R}^n, A(t) \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ and is real continuous for all $t \in \mathbf{R}$ and $\sup \|A(t)\| < \infty$.

Let $X(t)$ be a normal fundamental matrix of (8) and $\sigma_x = \sum\limits_{k=1}^{m} n_k\alpha_k$ be the sum of all its exponent numbers ([1]).

**Definition.** ([1]) The linear system (8) is said to be regular iff

$$\sigma_x = \varliminf_{t\to\infty} \frac{1}{t}\int\limits_{t_0}^{t} \operatorname{Sp} A(\tau)d\tau.$$

**Lemma.** *A necessary and sufficient condition that the system (8) to be regular one is there exists a generalized Lyapunov transformation carrying the system (8) to the system with constant matrix $B \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$:*

$$\frac{dy}{dt} = By \tag{9}$$

**Proof.** Let $y = L(t)x$ be a generalized Lyapunov transformation, $X(t)$ be a normal fundamental matrix of system (8). It follows that $Y(t) = L(t)X(t)$ is a fundamental matrix of system (9). Since

$$\det Y(t) = \det L(t) \det X(t),$$

we have

$$\det Y(t_0)\exp(t-t_0)\,\mathrm{Sp}\,B = \det L(t)\det X(t_0)\exp\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1$$

$$\exp\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1 = |c\,(t_0)|\left|\det L^{-1}(t)\right|\exp\left[(t-t_0)\,\mathrm{Sp}\,B\right],$$

$$\text{where}\quad c(t_0)=\det\left[Y(t_0)X^{-1}(t_0)\right],$$

$$\Rightarrow\quad \frac{1}{t}\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1 = \frac{1}{t}ln\,|c(t_0)| + \frac{1}{t}ln\left|\det L^{-1}(t)\right| + \left(1-\frac{t_0}{t}\right)\mathrm{Sp}\,B$$

$$\Rightarrow\quad \overline{\lim_{t\to\infty}}\,\frac{1}{t}\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1 = \mathrm{Sp}\,B + \chi\left[\det L^{-1}(t)\right].$$

Because of $\chi\left[L^{-1}(t)\right]=0$ we have

$$\chi\left[\det L^{-1}(t)\right]\le n\chi\left[L^{-1}(t)\right]=0$$

Analogously, from $\chi\left[L(t)\right]=0$ it follows that

$$\chi\left[\det L(t)\right]\le 0$$

On the other hand, since

$$\det L(t).\det L^{-1}(t)=0,$$

the following is held: $\chi\left[\det L(t)\right]+\chi\left[\det L^{-1}(t)\right]\ge 0$.

Therefore $\chi\left[\det L(t)\right]=\chi\left[\det L^{-1}(t)\right]=0$

It follows from these equalities that

$$\lim_{t\to\infty}\frac{1}{t}ln\left|\det L^{-1}(t)\right|=0$$

and finally

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_{t_0}^{t}\mathrm{Sp}\,A(t_1)dt_1=\mathrm{Sp}\,B.$$

Since the Lyapunov transformation conserves Lyapunov exponents and the $X$ is normal, $Y$ is normal too and

$$\sigma_x = \sigma_y = \operatorname{Sp} B,$$

we have

$$\sigma_x = \lim_{t\to\infty} \frac{1}{t}\int\limits_{t_0}^{t} \operatorname{Sp} A(t_1)dt_1,$$

i.e. the system (8) is regular.

Conversely, let the system (8) be regular. We will denote by $X(t)$ the fundamental normal matrix of (8), which has the exponent numbers: $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$. Consider the Jordan matrix $B$, in which $\lambda_1, \ldots, \lambda_n$ are elements on the diagonal.

Denoting $Y(t)$ the fundamental normal matrix of the system (9), we constate that the column of which has the same exponent numbers (with the same order): $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Putting $L(t) = Y(t)X^{-1}(t)$ we will prove that $y = L(t)x$ is a generalized Lyapunov transformation.

Suppose that

$$Y(t) = \begin{pmatrix} y_{11}(t) & y_{12}(t) & \ldots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \ldots & y_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \ldots & y_{nn}(t) \end{pmatrix}$$

$$X^{-1}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \ldots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \ldots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \ldots & x_{nn}(t) \end{pmatrix}.$$

Tdilehen $\chi\left[y^{(k)}\right] = \lambda_k$, where $y^{(k)}(t) = \operatorname{colon}(y_{1k}(t)\ldots y_{nk}(t))$.

Because of the regularity of (8), we have $\chi\left[x^{(k)}(t)\right] = -\lambda_k$, where $x^{(k)}(t) = (x_{k1}(t)\ldots x_{kn}(t))$.

We consider now the diagonal matrix

$$\Delta = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

We find then

$$L(t) = Y(t)e^{-t\Delta}e^{t\Delta}X^{-1}(t) = \phi(t)\Psi(t)$$

in which $\phi(t) = Y(t)e^{-t\Delta}, \Psi(t) = e^{t\Delta}X^{-1}(t)$. It follows that

$$\chi\left[\phi(t)\right] = \max_{j,k} \chi\left[y_{jk}(t)e^{-\lambda_k t}\right] = 0$$

$$\chi\left[\Psi(t)\right] = \max_{j,k} \chi\left[x_{j,k}(t)e^{\lambda_j t}\right] = 0.$$

Consequently,

$$\chi\left[L(t)\right] \leq \chi\left[\phi(t)\right] + \chi\left[\Psi(t)\right] = 0$$

Analogously we can prove that $\chi\left[L^{-1}(t)\right] \leq 0$.

However, from $L(t) \cdot L^{-1}(t) = E$, we immediately find that $\chi\left[L(t)\right] + \chi\left[L^{-1}(t)\right] \geq 0$, i.e. $\chi\left[L(t)\right] = \chi\left[L^{-1}(t)\right] = 0$. The lemma is proved.

**Definition.** A linear differential equation:

$$\frac{dx}{dt} = A(t)x, \tag{10}$$

where $A(t) \in \mathcal{L}(E,E)$ and is continuous for all $t \in \mathbf{R}$ and $\sup_t \|A(t)\| < \infty$, is said to be regular one iff there is a generalized Lyapunov transformation $y = L(t)x$ carrying which to the linear differential equation with constant operator:

$$\frac{dy}{dt} = By. \tag{11}$$

Now we shall give a main theorem to regular differential equations on Banach spaces.

Let consider differential equation

$$\frac{dx}{dt} = A(t)x + f(x,t), \tag{12}$$

where $A(t) \in \mathcal{L}(E,E)$ and $\sup_{t\in\mathbf{R}} \|A(t)\| < \infty, f \in C^{(1,0)}(E\times\mathbf{R}), f(0,t) \equiv 0,$

$$\|f(x,t)\| \leq \Psi(t)\|x\|^m \quad (m > 1); \quad \chi[\Psi(t)] = 0.$$

Under these conditions, we show the following theorem:

**Theorem.** *If the equation (10) is regular and all its characteristic exponents are not larger than $-\lambda < 0$, the trivial solution $x = 0$ of the equation (10) is exponential stability* ([7]). *I.e there exist $N > 0, A > 0$ such that*

$$\|x(t)\| \leq Ae^{-N(t-t_0)}\|x(t_0)\|$$

*for all solutions* $x(t)$ *of* (12).

**Proof.** We denote by $X(t)(X(t_0) = Id_E)$ its Cauchy operator of equation (10) ([7], p. 147).

**1.** First we will estimate the resolvant operator $K(t,\tau) = X(t)X^{-1}(\tau)$ $(t_0 \leq \tau \leq t)$.

Because of the regularity of the equation (10), there is a generalized Lyapunov transformation $y = L(t)x$ carrying equation (10) to equation (11).

We have $Y(t) = L(t)X(t)$ is resolvant operator of the equation (11).

If we put $H(t,\tau) = Y(t)Y^{-1}(\tau)$ then $K(t,\tau) = L(t)H(t,\tau)L^{-1}(\tau)$.

Suppose that all characteristic exponents of the equation (10) are not larger than $\alpha$.

Hence all those of the equation (11) are not too than $\alpha$, that is for every solution $y(t) = Y(t)y_0$ and $\varepsilon > 0$ there exists $c > 0$ we have

$$\|y(t)\| \leq ce^{(\alpha+\varepsilon/2)t}, \quad \forall t \geq t_0.$$

Then, the operator's family $\left\{e^{-(\alpha+\varepsilon/2)t}Y(t), t \geq t_0\right\}$ is point-bounded.

By virtue of the Banach–Steinhauss there exists $c_1, > 0$ such that:

$$\|e^{(-\alpha+\varepsilon/2)t}Y(t)\| \leq c_1 \Leftrightarrow \|Y(t)\| \leq c_1 e^{(\alpha+\varepsilon/2)t}.$$

Therefore $\|H(t,\tau)\| = \|Y(t-\tau\| \leq c_1 e^{(\alpha_\varepsilon/2)(t-\tau)}$ for the equation with constant operator (11)

On the other hand

$$\chi\left[L(t)\right] = \chi\left[L^{-1}(t)\right] = 0 \Leftrightarrow \begin{cases} \|L(t)\| \leq c_2 e^{\frac{\varepsilon}{2}t} \\ \|L^{-1}(\tau)\| \leq c_3 e^{\frac{\varepsilon}{2}\tau}. \end{cases}$$

It follows that

$$\|K(t,\tau)\| \leq \|L(t)\|\|H(t,\tau)\|\|L^{-1}(\tau)\|$$

(13)

$$\leq c_1c_2c_3e^{(\alpha+\varepsilon)(t-\tau)}e^{\varepsilon\tau} = c(\varepsilon,t_0)e^{(\alpha+\varepsilon)(t-\tau)}$$

where $c = c_1c_2c_3e^{-(\alpha+\varepsilon)\tau}$.

Since $K(t,t_0) = X(t)$ we have

(14) $$\|X(t)\| \leq ce^{(\alpha+\varepsilon)t}$$

In the case, when $\alpha < 0$, there exists a positive number $\varepsilon$ such that $\alpha + \varepsilon \leq 0$, whence

$$\|K(t,\tau)\| \leq ce^{\varepsilon t}, \quad \|X(t)\| \leq c. \tag{15}$$

**2.** We will now prove the theorem. Denoting

$$y = \chi e^{\gamma(t-t_0)} \tag{16}$$

where $\gamma$ is a positive number such that $0 < \gamma < \lambda$, the equation (12) will be transformed to:

$$\frac{dy}{dt} = B(t)y + g(t,y) \tag{17}$$

with $B(t) = A(t) + \gamma Id_E$

$$g(t,y) = \exp(\gamma(t-t_0))f\left(t, ye^{-\gamma(t-t_0)}\right). \tag{18}$$

Now we show that the equation

$$\frac{d\eta}{dt} = B(t)\eta \tag{19}$$

is regular. Indeed, by the regularity of (10) there is a generalized Lyapunov transformation $z = L(t)x$ carrying (10) to the equation with constant operator:

$$\frac{dz}{dt} = Cz$$

where

$$C = L'(t)L^{-1}(t) + L(t)A(t)L^{-1}(t).$$

The transformation $\xi = L(t)\eta$ implies the following:

$$\frac{d\xi}{dt} = \left[L'(t)L^{-1}(t) + L(t)B(t)L^{-1}(t)\right]\xi = (C + \gamma Id_E)\,\xi.$$

The regularity of (19) is proved.

We denote by $\eta(t)$ the solution of (19) and then $e^{-\gamma(t-t_0)}\eta(t)$ is the solution of (10).

This implies:

$$\chi\left[\eta(t)e^{-\gamma(t-t_0)}\right] \leq -\lambda$$

$$\Rightarrow \chi\left[\eta(t)\right] \leq \chi\left[e^{\gamma(t-t_0)}\right] + \chi\left[\eta(t)e^{-\gamma(t-t_0)}\right] \leq -\lambda + \gamma < 0.$$

By virtue of the estimation of the resolvant operator the following inequality is true:

$$\|K(t,\tau)\| \leq Ne^{\varepsilon\tau}; \quad t_0 \leq \tau < \infty,$$

where $K(t,\tau)$ is the resolvant operator of (10).

Now considerint the solution of (17)

$$y(t) = K(t,t_0)y(t_0) + \int_{t_0}^{t} K(t,\tau)g(\tau,y(\tau))d\tau,$$

we have

$$\|y(t)\| \leq \|K(t,t_0)\| \cdot \|y(t_0)\| + \int_{t_0}^{t} \|K(t,\tau)\| \cdot \|g(\tau,y(\tau))\| d\tau$$

$$\leq Ne^{\varepsilon(t_0)}\|y(t_0)\| + \int_{t_0}^{t} Ne^{\varepsilon\tau}e^{\gamma(\tau-t_0)}\Psi(\tau)\|y(\tau)\|^m e^{-m\gamma(\tau-t_0)}d\tau$$

$$\leq Ne^{\varepsilon t_0}\|y(t_0)\| + \int_{t_0}^{t} Ne^{\varepsilon\tau}e^{(1-m)\gamma(\tau-t_0)}ce^{\varepsilon\tau}\|y(\tau)\|^m d\tau$$

$$= c_1\|y(t_0)\| + \int_{t_0}^{t} c_2 e^{[2\varepsilon-(m-1)\gamma](\tau-t_0)}\|y(\tau)\|^m d\tau$$

where $c_1 = Ne^{\varepsilon t_0}, c_2 = cNe^{-2\varepsilon t_0}$.

Hence

$$\|y(t)\| \leq c_1\|y(t_0)\| + \int_{t_0}^{t} c_2 e^{-\delta(\tau-t_0)}\|y(\tau)\|^m d\tau, \tag{20}$$

where $\delta = (m-1)\gamma - 2\varepsilon$.

We will find the positive number $\varepsilon$ such that $\delta > 0$.

Since

$$\int_{t_0}^{t} e^{-\delta(\tau-t_0)}d\tau = \frac{1}{\delta} - \frac{1}{\delta}e^{-\delta(t-t_0)} < \frac{1}{\delta}$$

there is $\Delta > 0$ such that

$$N = (m-1)c_1^{m-1}\|y(t_0)\|^{m-1}\int\limits_{t_0}^{t} c_2 e^{-\delta(\tau - t_0)}d\tau < 1$$

provided that

$$\|y(t_0)\| < \Delta.$$

We apply here the lemma of Bihari [8] and find

$$\|y(t)\| \leq \frac{c_1\|y(t_0)\|}{[1-N]^{\frac{1}{m-1}}} = A\|y(t_0)\|, A = \frac{c_1}{[1-N]^{\frac{1}{m-1}}}$$

$$\Rightarrow \|x(t)\| \leq Ae^{-\gamma(t-t_0)}\|x(t_0)\|, (x(t_0) = y(t_0))$$

that is the exponential stability of the solution $x = 0$ of (12), and the proof of the theorem is finished.

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